

Phase Structure of Thermal QCD/QED: A Gauge Invariant Solution of the HTL Resummed Improved Ladder Dyson-Schwinger Equation

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Abstract: Based on the hard-thermal-loop resummed improved ladder Dyson-Schwinger equation for the fermion mass function, we propose a procedure how we can get the gauge invariant solution in the sense it satisfies the Ward-Takahashi identity. Results of the numerical analysis are shown and properties of the “gauge-invariant” solutions are discussed.

1. Introduction and Summary

The Dyson-Schwinger equation (DSE) has proven itself a powerful tool to investigate *with the analytic procedure* the nonperturbative structure of field theories, such as the chiral phase transition of gauge theories. Actually analyses based on the DSE have successfully clarified the phase structure of vacuum QEC/QCD [1,2,3]. Here we must take note of the fact that these DSE analyses in vacuum gauge theories were carried out in the Landau gauge with the ladder interaction kernel. Same analyses have been performed at finite temperature(T) and/or density(μ) also in the Landau gauge with the ladder interaction kernel [4,5,6], and with the hard-thermal-loop (HTL) resummed improved ladder interaction kernel [7].

The reason why the ladder approximation is used is that the full DSEs are coupled integral equations for several unknown functions, thus are hard to be solved without introducing appropriate approximations. We usually adopt the step-by-step approach to this problem, firstly approximate the integration kernel by the tree, or, ladder interaction kernel, next use the improved ladder one, etc. The possibility of such a systematic improvement through the well-established analytic procedure is one of the important characteristics of the DSE. In fact, results of Ref.[7] show that at finite temperature/density it is important to correctly take the dominant thermal effect into the interaction kernel in terms of the HTL resummation.

In case of the vacuum QED in the Landau gauge DSE with the ladder kernel for the fermion mass function, the fermion wave function renormalization constant is guaranteed to be unity [1], namely the Ward-Takahashi identity is automatically satisfied. Thus irrespective of the problem of the ladder approximation, the results obtained would be gauge invariant.

At finite T and/or μ , however, there is no such guarantee. In fact, even in the Landau gauge the fermion wave function renormalization constant largely deviates from unity [7,8], being not even real. At finite T and/or μ the results obtained from the ladder DSE explicitly violate the Ward-Takahashi identity, thus depend on the choice of gauge. All the preceding analyses [4-7] suffer from the gauge-dependence problem coming from the ladder approximation of the interaction kernel, their physical meaning being obscure.

In this paper, we present, in the analysis of the HTL resummed improved ladder DS equation for the fermion mass function in thermal QED/QCD, the procedure to get the gauge invariant solution in the sense it satisfies the Ward-Takahashi identity. We firstly show that the solutions of the HTL resummed improved ladder DS equation in thermal QED/QCD suffer from the problem of gauge-parameter dependence, then solve numerically the DSE constrained to satisfy the Ward-Takahashi identity and investigate the properties of the “gauge invariant” solution.

Part of the preliminary result of the analysis was reported in Ref.[9], showing the effectiveness of the procedure.

Results of the present analysis are summarized as follows:

- (1) The solution of the HTL resummed improved ladder DS equation depends strongly on the choice of the gauge parameter within the momentum-independent gauge. This type of solution always shows the explicit contradiction with the Ward-Takahashi identity.
- (2) We can determine numerically the solution that satisfies the Ward-Takahashi identity, namely the solution in which the fermion wave function renormalization constant is almost equal to unity. To get such a solution it is essential that we work in the nonlinear gauge where the gauge parameter ξ depends on the momentum of the gauge boson.
- (3) The chiral phase transition in the massless thermal QED/QCD is confirmed to occur through the second order transition; a dynamical fermion mass is generated at the critical temperature or at the critical coupling constant without discontinuity.
- (4) The effect of thermal fluctuation on the chiral symmetry breaking and/or restoration is smaller than that expected in the previous analysis in the Landau gauge [7].

2. DS equation for the fermion self-energy function Σ_R

2-1. DS equation in the HTL resummed improved ladder approximation

The fermion self-energy function Σ_R appearing in the fermion propagator S_R

$$S_R(P) = [\not{P} + i\epsilon\gamma^0 - \Sigma_R(P)]^{-1} \quad (1)$$

can be decomposed at finite temperature and/or density as

$$\Sigma_R(P) = (1 - A(P))\not{P}\gamma^i - B(P)\gamma^0 + C(P) \quad (2)$$

with $A(P)$, $B(P)$ and $C(P)$ being the three scalar invariants to be determined. In the present analysis, we use the HTL resummed form ${}^*G_{\mu\nu}$ for the gauge boson propagator,

$${}^*G^{\mu\nu}(K) = \frac{1}{{}^*\Pi_T - K^2 - i\epsilon k_0} A^{\mu\nu} + \frac{1}{{}^*\Pi_L - K^2 - i\epsilon k_0} B^{\mu\nu} - \frac{\xi}{K^2 + i\epsilon k_0} D^{\mu\nu}, \quad (3)$$

where ${}^*\Pi_{L/T}$ is the HTL resummed longitudinal/transverse photon self-energy function [10]. $A^{\mu\nu}$, $B^{\mu\nu}$ and $D^{\mu\nu}$ are the projection tensors [11],

$$A^{\mu\nu} = g^{\mu\nu} - B^{\mu\nu} - D^{\mu\nu}, \quad (4)$$

$$B^{\mu\nu} = -\tilde{K}^\mu \tilde{K}^\nu / K^2, \quad (5)$$

$$D^{\mu\nu} = K^\mu K^\nu / K^2, \quad (6)$$

where $\tilde{K} = (k, k_0 \hat{\mathbf{k}})$, $k = \sqrt{\mathbf{k}^2}$ and $\hat{\mathbf{k}} = \mathbf{k}/k$ denotes the unit three vector along \mathbf{k} .

The parameter ξ appearing in the term proportional to the projection tensor $D_{\mu\nu}$ represents the gauge-fixing parameter ($\xi = 0$ in the Landau gauge). This gauge term plays an important role in the present analysis.

The vertex function is approximated by the tree (point) vertex. With the instantaneous exchange approximation for the longitudinal gauge boson propagator, we get the DSEs for the three invariant functions $A(P)$, $B(P)$ and $C(P)$

$$\begin{aligned}
-p^2[1 - A(P)] = & -e^2 \int \frac{d^4 K}{(2\pi)^4} \left[\{1 + 2n_B(p_0 - k_0)\} \text{Im} [{}^*G_R^{\rho\sigma}(P - K)] \times \right. \\
& \left[\{K_\sigma P_\rho + K_\rho P_\sigma - p_0(K_\sigma g_{\rho 0} + K_\rho g_{\sigma 0}) - k_0(P_\sigma g_{\rho 0} + P_\rho g_{\sigma 0}) + pkz g_{\sigma\rho} \right. \\
& + 2p_0 k_0 g_{\sigma 0} g_{\rho 0} \} \frac{A(K)}{[k_0 + B(K) + i\epsilon]^2 - A(K)^2 k^2 - C(K)^2} + \{P_\sigma g_{\rho 0} \\
& + P_\rho g_{\sigma 0} - 2p_0 g_{\sigma 0} g_{\rho 0} \} \frac{k_0 + B(K)}{[k_0 + B(K) + i\epsilon]^2 - A(K)^2 k^2 - C(K)^2} \Big] \\
& + \{1 - 2n_F(k_0)\} {}^*G_R^{\rho\sigma}(P - K) \text{Im} \left[\{K_\sigma P_\rho + K_\rho P_\sigma - p_0(K_\sigma g_{\rho 0} + K_\rho g_{\sigma 0}) \right. \\
& - k_0(P_\sigma g_{\rho 0} + P_\rho g_{\sigma 0}) + pkz g_{\sigma\rho} + 2p_0 k_0 g_{\sigma 0} g_{\rho 0} \} \times \\
& \frac{A(K)}{[k_0 + B(K) + i\epsilon]^2 - A(K)^2 k^2 - C(K)^2} + \{P_\sigma g_{\rho 0} + P_\rho g_{\sigma 0} \\
& - 2p_0 g_{\sigma 0} g_{\rho 0} \} \frac{k_0 + B(K)}{[k_0 + B(K) + i\epsilon]^2 - A(K)^2 k^2 - C(K)^2} \Big] \Big] , \tag{7}
\end{aligned}$$

$$\begin{aligned}
-B(P) = & -e^2 \int \frac{d^4 K}{(2\pi)^4} \left[\{1 + 2n_B(p_0 - k_0)\} \text{Im} [{}^*G_R^{\rho\sigma}(P - K)] \times \right. \\
& \left[\{K_\sigma g_{\rho 0} + K_\rho g_{\sigma 0} - 2k_0 g_{\sigma 0} g_{\rho 0} \} \frac{A(K)}{[k_0 + B(K) + i\epsilon]^2 - A(K)^2 k^2 - C(K)^2} \right. \\
& + \{2g_{\rho 0} 2g_{\sigma 0} - g_{\sigma\rho} \} \frac{k_0 + B(K)}{[k_0 + B(K) + i\epsilon]^2 - A(K)^2 k^2 - C(K)^2} \Big] \\
& + \{1 - 2n_F(k_0)\} {}^*G_R^{\rho\sigma}(P - K) \text{Im} \left[\frac{A(K)}{[k_0 + B(K) + i\epsilon]^2 - A(K)^2 k^2 - C(K)^2} \right. \\
& \times \{K_\sigma g_{\rho 0} + K_\rho g_{\sigma 0} - 2k_0 g_{\sigma 0} g_{\rho 0} \} \\
& + \frac{k_0 + B(K)}{[k_0 + B(K) + i\epsilon]^2 - A(K)^2 k^2 - C(K)^2} \{2g_{\rho 0} 2g_{\sigma 0} - g_{\sigma\rho} \} \Big] \Big] , \tag{8}
\end{aligned}$$

$$\begin{aligned}
C(P) = & -e^2 \int \frac{d^4 K}{(2\pi)^4} g_{\sigma\rho} \{1 + 2n_B(p_0 - k_0)\} \text{Im} [{}^*G_R^{\rho\sigma}(P - K)] \times \\
& \left[\frac{C(K)}{[k_0 + B(K) + i\epsilon]^2 - A(K)^2 k^2 - C(K)^2} + \{1 - 2n_F(k_0)\} \times \right. \\
& \left. {}^*G_R^{\rho\sigma}(P - K) \text{Im} \left[\frac{C(K)}{[k_0 + B(K) + i\epsilon]^2 - A(K)^2 k^2 - C(K)^2} \right] \right] . \tag{9}
\end{aligned}$$

Above DSEs may have several solutions, and we choose the “true” solution by evaluating the effective potential $V[S_R]$ for the fermion propagator function S_R , then finding the lowest energy solution.

$$\begin{aligned}
V[S_R] = & i\text{Tr} [\not{P} S_R] + i\text{Tr} \ln [iS_R^{-1}] \\
& - \frac{e^2}{2} \int \frac{d^4 K}{(2\pi)^4} \int \frac{d^4 P}{(2\pi)^4} \frac{1}{2} \text{tr} [\gamma_\mu S_R(K) \gamma_\nu S_R(P) D_C^{\mu\nu}(P - K) \\
& + \gamma_\mu S_C(K) \gamma_\nu S_R(P) D_R^{\mu\nu}(P - K) + \gamma_\mu S_R(K) \gamma_\nu S_C(P) D_A^{\mu\nu}(P - K)] , \tag{10}
\end{aligned}$$

2-2. Procedure to get the “gauge-invariant” solution

The function $A(P)$ above is nothing but the inverse of the fermion wave function renormalization constant Z_2 , thus must be unity in order to satisfy the Ward-Takahashi identity in the ladder DSE analysis, where the vertex function receives no renormalization effect, $Z_1 = 1$.

We must solve the above DSEs and get the solution satisfying the Ward-Takahashi identity $Z_2 = Z_1 (= 1)$, where $Z_2 = A(P)^{-1}$. The procedure to get the “gauge invariant” solution is as follows;

(1) Assume the nonlinear gauge such that the gauge parameter ξ being a function of the momentum $K = (k_0, k)$ carried by the gauge boson. We parametrize ξ as

$$\xi(k_0, k) = \sum \xi_{mn} H_m(k_0) L_n(k), \quad k = \sqrt{\mathbf{k}^2}, \quad (11)$$

where ξ_{mn} are unknown parameters to be determined. H_m and L_n can in general be any orthonormal functions, and we here take the Hermite functions for H_m and the Laguerre functions for L_n .

(2) When solving the above DSEs iteratively, impose the condition $A(P) = 1$ by constraint for the input-functions at each step of the iteration.

(3) Determine ξ_{mn} so as to minimize $|A(P) - 1|^2$ for the output-functions and find the solutions for $B(P)$ and $C(P)$.

3. “Gauge invariant” solution consistent with the Ward-Takahashi identity

Here we present the results obtained by the momentum-dependent gauge parameter ξ . Number of parameters ξ_{mn} to minimize $|A(P) - 1|^2$ is $2 \times 3 \times 2 = 12$ (i.e., $m = 0 \sim 2$ and $n = 0, 1$) in the case of complex ξ , and $4 \times 3 = 12$ (i.e., $m = 0 \sim 3$ and $n = 0 \sim 2$) in the case of real ξ . All the quantities with the mass dimension are evaluated in the unit of Λ , the cut-off parameter introduced as usual to regularize the DSEs.

Before presenting the “gauge invariant” solution, we show in Fig.1 the result of the critical temperature analysis for several values of constant ξ to get a rough image for the size of gauge dependence. As can be seen clearly the critical temperature strongly depends on the gauge, but the order of the phase transition does not.

Now we present the solution consistent with the Ward-Takahashi identity, i.e., the “gauge invariant” solution. Analysis is now in progress, and the results shown below are, at present, still preliminary.

Firstly in Fig.2 we show $Re[A(P)]$. For comparison, results in the constant ξ analyses are also shown in the same figure.

Next let us study the property of the phase transition. Fig.3 shows the real part of the fermion mass $Re[M(P)]$, $M(P) \equiv C(P)/A(P)$ ($= C(P)$, because $A(P) = 1$), obtained from the “gauge invariant” solution, as a function of the temperature T . The mass is evaluated at $p_0 = 0$, $p = 0.1$, to be consistent with the standard prescription to define the mass in the static limit, $p_0 = 0$, $p \rightarrow 0$.

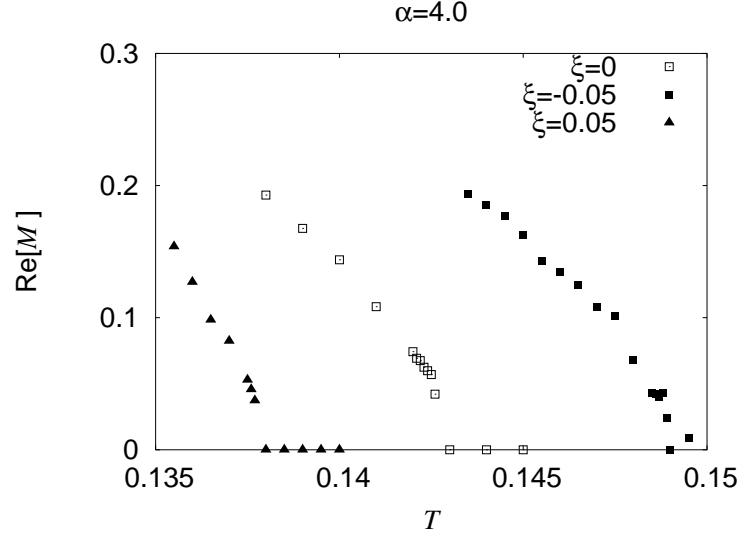


Figure 1: Gauge-parameter-dependence of the fermion mass $\text{Re}[M]$ at the coupling constant $\alpha = 4.0$ evaluated at $p_0 = 0$, $p = 0.1$.

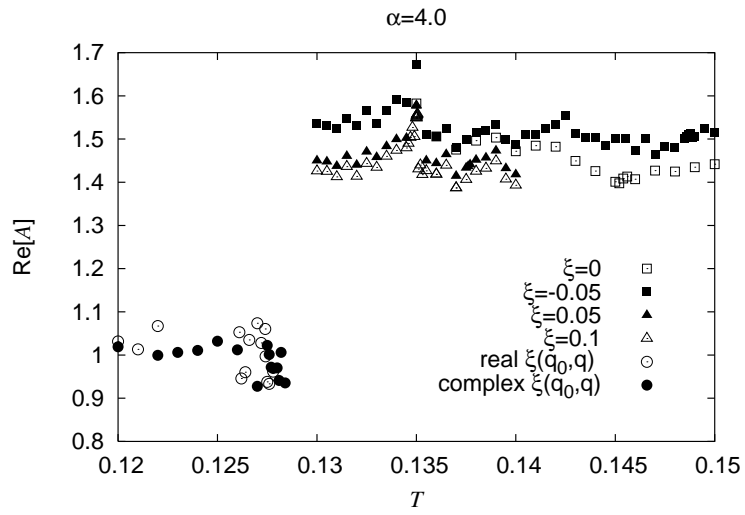


Figure 2: Comparison of the wave function renormalization constant $\text{Re}[A]$ at the coupling constant $\alpha = 4.0$ evaluated at $p_0 = 0$, $p = 0.1$.

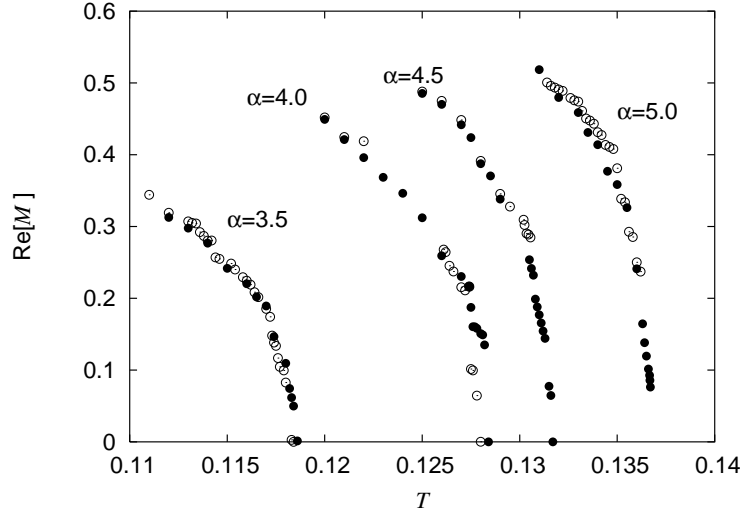


Figure 3: Temperature-dependence of the fermion mass $\text{Re}[M]$ for various values of the coupling constant $\alpha =$ evaluated at $p_0 = 0$, $p = 0.1$, see text. Open circle denotes real ξ data, while solid circle denotes complex ξ data.

Analyses to determine the critical temperature T_c , the critical coupling α_c , and two critical exponents ν and η are now in progress.

As can be clearly seen in the above Figures 2 and 3, two solutions obtained in the different two prescriptions, complex v.s. real gauge parameters ξ , show complete agreement. This fact indicates that the solution obtained in the present procedure does not depend on the choice of the gauge parameter, namely that the solution is “gauge invariant”. The results shown in this paper are still preliminary, and we will soon report the results of full analysis.

The phase boundary curve in the (T, α) -plane thus determined shows that the region of the symmetry broken phase shrinks to the low-temperature and the strong-coupling side compared with that of the Landau gauge. This fact means that the effect of thermal fluctuation on the chiral symmetry breaking/restoration is smaller than that expected in the previous analysis in the Landau gauge [7].

4. Discussion and comments

Results presented in the present paper are still preliminary, because of the rough analysis of the data processing. We are now refining the data analysis and soon get the results of the thorough reanalysis. Though the main conclusion will not be altered, several important remarks should be added.

(1) We performed the present analysis in two prescriptions for the nonlinear gauge parameter ξ , complex v.s. real. Allowing the gauge parameter ξ to be a complex value may correspond to studying the non-hermite dynamics, thus may cause some troubles. In this sense we are interested in the result obtained by restricting the gauge parameter to the real value. What we found is a remarkable result: In both cases results completely agree, thus getting a solution totally independent of the choice of gauges. This fact strongly indicates that we can get the gauge-invariant physical result by studying the DSE with the ladder interaction kernel through

the present procedure.

(2) In the present analysis, the consistency of the solution with the Ward-Takahashi identity is respected only by imposing the condition $A(P) \approx 1$. Needless to say, in solving the (improved) ladder Dyson-Schwinger equation, there are no solutions totally consistent with the Ward-Takahashi identity. Despite this fact, following point should be closely examined: At least around or in the static limit, $p_0 = 0$, $p \rightarrow 0$, where we calculated (defined) the mass, each invariant function $A(P)$, $B(P)$ or $C(P)$ should not have big momentum dependence. This condition may be important in connection with the consistency of the obtained solution with the gauge invariance. Result of the present analysis shows that at least $B(P)$ and $C(P)$ satisfy this condition.

References and footnotes

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